Abstract

The exploitation of the mean-reversion of commodity prices is important for inventory management, inflation forecasting and contingent claim pricing. Bessembinder, Coughenour, Seguin and Smoller (1995) document the mean-reversion of commodity spot prices using futures term structure data; however, mean-reversion to a constant level is rejected in nearly all studies using historical spot price time series. This indicates that the spot prices revert to a stochastic long-run mean. By recognizing this, I propose a reduced-form model with the stochastic long-run mean as a separate factor. This model fits the futures dynamics better than do classical models such as the Gibson-Schwartz (1990) model and the Casassus-Collin-Dufresne (2005) model with a constant interest rate. An application for option pricing is also presented in this paper.
1 Introduction

Finding mean-reversion in commodity spot prices has important practical implications. For instance, having returns associated with mean-reversion implies that return variances do not increase linearly with time. Casassus and Collin-Dufresne (CC 2005) show that if commodity prices revert to a constant mean, the prices of options on commodity futures will be significantly smaller than in the random walk case. In addition, the failure to identify the mean-reversion of commodity prices may influence values of real options, such as valuations of mines and inventories.\footnote{The option of holding inventory is an option on the spread of futures with different maturities.} Moreover, the mean-reversion of spot prices will also impact monetary and fiscal policy, as expected commodity prices highly influence households’ expectations regarding inflation and hence also influence consumption levels.\footnote{The "energy tax" in Bernanke (2004, 2006) is along this line.} However, the question of whether commodity prices revert to a constant mean is still under debate.

Bessembinder, Coughenour, Seguin, and Smoller (BCSS 1995) argue in favor of mean-reversion in commodity spot prices. Specifically, the authors discover a negative relationship between the (log) slope of a commodity futures term structure and the corresponding commodity spot price, which shows that the log-spot price is mean-reverting in the risk-neutral measure.\footnote{Note that in the risk-neutral measure, the spot commodity prices can be mean-reverting, however the futures prices with fixed maturities are martingales.} Note the BCSS test does not require the log-spot price to revert to a constant mean. Moreover, BCSS also discover a negative relationship between the risk premium of a spot price and the (log) spot price, which is also confirmed by empirical findings in this paper using oil and copper futures data. Some contingent claim pricing papers on commodity futures employ a mean-reverting process to a constant level to model the (log) spot commodity prices; among them are the one-factor model in Schwartz (1997), Geman and Nguyen (2005) and CC(2005). These models can generate the negative relationship between the slope of futures’ term structure and the spot price.

Statistically, if the log-spot price reverts to a constant long-run mean in the risk-neutral measure, and if the risk premium is negatively related to the log-spot price, the log of the commodity
price must revert to a constant mean in the physical measure with a even stronger speed than that in the risk-neutral measure. However, this reversion to a constant mean is commonly rejected by many researchers. For example, Cuddington and Urzua (1988), Gersovitz and Paxson (1990) and Dempster, Medova and Tang (2008) document that commodity prices do not revert to a constant mean, as it is quite hard to statistically reject the most parsimonious random walk model using historical time series. Cashin, Liang and McDermott (2000) have shown that shocks to commodity prices are typically persistent,\(^4\) while Grilli and Yang (1988), using a dataset from 1900 to 1986, find that real primary commodity prices have a trend of about 0.5 percent a year. Alquist and Kilian (2008) argue that no approach is superior to the use of simple Brownian motion for forecasting oil prices. Bernard, Khalaf, Kichian and McMahon (2008) find that the Schwartz-Smith model (which is the same as Gibson and Schwartz (1990) model) are superior to geometric Brownian motion models in forecasting aluminum prices. Gibson and Schwartz (GS, 1990) model and the three-factor model in Schwartz (1997) utilize geometric Brownian motion to model commodity (log) spot prices. Hence models in Gibson-Schwartz (GS, 1990) and Schwartz-Smith (2000) cannot generate the negative relationship between the slope of the futures term structure and spot prices discovered by BCSS.

The above-mentioned discrepancy implies that the commodity price reverts to a time-varying long-run mean. If the long-run mean is stochastic and non-stationary in the physical measure, all mean-reversion tests for a constant mean will reject the mean-reversion hypothesis using historical time series. Moreover, since commodity prices do revert to their stochastic long-run mean, the negative relationship should be discovered between the commodity spot price and the slope of the futures term structure as shown in BCSS. Therefore, the notion of the time-varying long-run mean is consistent with the test by BCSS and many tests based on historical commodity time series.

Economically, since we all model the nominal dollar commodity prices instead of real prices, accumulated inflation, which is normally regarded as a non-stationary process, can be considered as a potential source of the stochastic long-run mean. Moreover, as commodities are usually

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\(^4\)Cashin, Liang and McDermott (2000) use the median-unbiased estimator to do the empirical test, which is superior to the results emanating from standard unit root regression.
traded globally, their fundamental values are largely related to the US dollar value, whose process is also normally regarded as a non-stationary process (Garman and Kohlhagen, 1983). Thus, this US dollar factor is also likely to make commodity prices revert to a stochastic mean. As shown in Greenspan (2004) and Bernanke (2004), long-term futures prices can be used to obtain the expectation of future spot prices. Figure 1 shows the evolution of nearby and long-term (22-month) oil futures contracts, where the long-term futures prices are apparently not a constant, which shows that investors do have different expectations of commodity prices over time.

To incorporate this notion into the modeling of futures prices, I build a three-factor model that includes the stochastic long-run mean, the log-spot price and convenience yield factors. The likelihood ratio and out-of-sample tests show that this model performs better than those without the stochastic long-run mean factor such as CC(2005) with a constant interest rate and GS(1990). Additionally, this model is also a test for the existence of a stochastic long-run mean using the whole historical futures term structures as inputs; if the volatility of the stochastic long-run mean factor is very small (or zero) for a certain commodity, its prices are likely to revert to a constant long-run mean.

The remainder of this paper is organized as follows. Section 2 proposes the idea of a stochastic long-run mean. Section 3 proposes a reduced-form model for commodity futures prices and also outlines a futures pricing formula. Section 4 shows calibration methods and results using oil and copper data. Section 5 shows an option pricing application of this model, and Section 6 concludes.

2 Stochastic Long-run Mean of Commodity Prices

As a preliminary step, using oil and copper futures data, I repeat the test conducted by BCSS on the relation between the spot and the slope of futures term structures and confirm a negative relationship between them. Additionally, BCSS also find that the risk premia associated with commodities decrease in their spot prices.\(^5\) I redo the same risk premium-spot test and report a significantly negative relationship that is consistent with BCSS (1995). I here omit the results

\(^5\)The t-statistics are not significant in BCSS.
for briefness; the complete details can be provided on request. As mentioned in the introduction, the fact that commodity prices do not revert to a constant mean in the physical measure together with the tests in BCSS suggests a time-varying long-run mean of commodity (log) spot prices. If the long-run mean is non-stationary, all mean-reversion tests for a constant long-run mean (such as the Augmented Dickey-Fuller test) should reject the mean reversion of spot prices, as the spot price is also non-stationary. For example, suppose the log-spot price $x$ reverts to its non-stationary long-run mean $y$ by:

\begin{align*}
    x_t - x_{t-1} &= \phi (y_{t-1} - x_{t-1}) + \varepsilon_1 \\
y_t - y_{t-1} &= \alpha + \varepsilon_2
\end{align*}

Hamilton (1994) shows that in (1) a cointegration relationship exists between the $x$ and $y$ factors and that both $x$ and $y$ factors are non-stationary.

Moreover, one substantial feature of the futures market is that it is a useful place to learn about informed opinions. Long-term futures prices can be used to obtain expectations of future spot prices (see Greenspan 2004 and Bernanke 2004). Figure 1 shows that the prices of long-term futures contracts significantly increase after 2002. This informs that investors do have different expectations over time. This makes perfect sense; as information about the demand and supply of a certain commodity (such as the demand data announced by the International Energy Agency) becomes known, market participants will revise their projections of commodity prices accordingly. Hence, the expectation of commodity prices is time-varying; for example, many financial institutions announce different forecasts of oil prices corresponding to different time horizons. I perform a simple augmented Dickey-Fuller test for the long-term futures prices, and the results cannot reject the hypothesis of non-stationary. The result is omitted for the sake of brevity; the complete details will be given based on requests.

Moreover, several "over-shooting models" predict that commodity prices tend to deviate moderately from their time-varying fundamentals and eventually readjust to meet them again. For
example, Frankel (1986) shows that a shift in monetary policy will cause the overshooting of commodity prices. Alquist and Kilian (2008) report that the increase in the uncertainty of the future supply of a commodity will also cause expectations to overshoot the actual commodity price. The overshooting models predict a time-varying expectation of future commodity prices and the negative relationship between the spot price and the slope of the futures term structure.

Since only futures prices, which are "derivatives", are observable in the markets, a reduced-form model is necessary to model them. In the next section, I propose a model with the feature of stochastic long-run mean of spot prices. The likelihood ratio and out-of-sample tests can examine the performance of my model relative to others; additionally, it can also test the time-varying long-run mean against a constant long-run mean.

3 The Model

In this paper, I propose a three-factor affine model where the log-spot price fluctuates around its stochastic long-run mean. Hence, I name the model the Stochastic Long-run Mean (SLM) model.

Define $x_t := \ln (S_t)$ as the log of the commodity spot price. Thus, in the risk-neutral measure,
the three factors governing the movement of the futures term structure are:

\[ dx_t = [a(y_t - x_t) - z_t] \, dt + \sigma_1 dW^Q_1 \]
\[ dy_t = u \, dt + \sigma_2 dW^Q_2 \]
\[ dz_t = (\phi - k z_t) \, dt + \sigma_3 dW^Q_3, \]

where \( W^Q_1, W^Q_2 \) and \( W^Q_3 \) are Brownian motions in the risk-neutral measure.

From this model, the log-spot price \( x_t \) fluctuates around \( y_t \) with mean-reversion speed \( a \). Consistent with the empirical facts mentioned in previous sections, I model the long-run mean \( y \) factor using a Brownian motion. The \( y \) factor is regarded as the permanent factor, as it has permanent shocks; in contrast, \( z \) is a temporary factor (a mean-reverting factor). From Hamilton (1994), both the \( x \) and \( y \) factors are non-stationary, and there exists a cointegration relationship between \( y_t \) and \( x_t \) with cointegration vector \([1, -1]\). We expect that \( a > 0 \), so the drift of \( x \) decreases in the \( x \) factor, i.e., the present deviation of the commodity prices from its long-run mean tends to decrease in the next period. This model reflects that the (log) spot price \( x \) reverts to a long-run mean \( y \) and that the long-run mean factor \( y \) is non-stationary.

In the risk-neutral measure, the drift of the spot price \( S_t \) has to satisfy \( E^Q_t [dS_t] / dt = (r - \delta_t)S_t \) where \( \delta_t \) denotes the net convenience yield of the commodity. Hence, the following relationship needs to be satisfied

\[ r - \delta_t - \frac{1}{2} \sigma_1^2 = a(y_t - x_t) - z_t, \]

or

\[ \delta_t = r - \frac{1}{2} \sigma_1^2 - a(y_t - x_t) + z_t, \]

if \( a > 0 \), the convenience yield has a positive relationship with the spot price \( x_t \). Note that the convenience yields’ dependence on the log-spot price is also one of the main features of CC(2005). CC(2005) also finds empirical support for this dependence. Since \( y_t \) and \( x_t \) are cointegrated,
although both $x_t$ and $y_t$ are not stationary, $y_t - x_t$ is stationary and hence in (3) the convenience yield $\delta_t$ is stationary. Note that both Schwartz (1997) and CC(2005) include a stochastic interest rate factor in modeling the futures term structure; however, as many scholars have pointed out (Miltersen, 2003 and Carmona and Ludkovski, 2004), the stochastic interest rate does not play a substantial role in the modeling of futures due to its relatively small volatility, especially when the time to maturity is relatively short. Hence, although it is not difficult to include an extra fourth factor–the stochastic interest rates–I do not include it in this paper for the sake of parsimony.

Assuming the risk premium $\lambda := [\lambda_x, \lambda_y, \lambda_z]^{\top}$ for $x, y$ and $z$ factors is expressed as

$$
\begin{align*}
\lambda_x &= \frac{1}{\sigma_1} \left( \tilde{\theta} - (\tilde{\alpha} - a) x_t \right) \\
\lambda_y &= \frac{1}{\sigma_2} (\tilde{u} - u) \\
\lambda_z &= \frac{1}{\sigma_3} (\tilde{\phi} - \phi).
\end{align*}
$$

I expect that $a < \tilde{\alpha}$, which corresponds to a negative relationship between the risk premium and the spot price. For the risk premia of $y$ and $z$ factors, I assume that they are constants for simplicity.

In the physical measure, the three factors change to

$$
\begin{align*}
\text{d}x_t &= \left( a y_t - \tilde{\alpha} x_t + \tilde{\theta} - z_t \right) \text{d}t + \sigma_1 \text{d}W_1^P \\
\text{d}y_t &= \tilde{u} \text{d}t + \sigma_2 \text{d}W_2^P \\
\text{d}z_t &= (\tilde{\phi} - k z_t) \text{d}t + \sigma_3 \text{d}W_3^P,
\end{align*}
$$

where $W_1^P, W_2^P$ and $W_3^P$ are Brownian motions in the physical measure.

Since our model belongs to the general exponential affine model as shown in Duffie, Pan and Singleton (2000), defining $\tau := T - t$, the futures prices $F(x, y, z, \tau)$ can be expressed analytically as

$$
F(x_t, y_t, z_t, \tau) = \exp \left[ A(\tau) x + B(\tau) y + C(\tau) z + D(\tau) \right]
$$

8
where

\[ A(\tau) = e^{-a\tau} \]
\[ B(\tau) = 1 - e^{-a\tau} \]
\[ C(\tau) = \frac{1}{k-a} (e^{-k\tau} - e^{-a\tau}) \]
\[ D(\tau) = u \left[ \tau - \frac{1 - e^{-a\tau}}{a} \right] + \phi \left[ \frac{1 - e^{-k\tau}}{k} - \frac{1 - e^{-a\tau}}{a} \right] + \frac{\sigma_1^2}{4} \frac{1 - e^{-2a\tau}}{a} \\
+ \frac{\sigma_2^2}{2} \left[ \tau - \frac{2(1 - e^{-a\tau})}{a} \right] + \frac{\sigma_3^2}{2(k-a)^2} \left[ \frac{1 - e^{-2k\tau}}{2k} - \frac{2(1 - e^{-(a+k)\tau})}{a+k} + \frac{1 - e^{-2a\tau}}{2a} \right] \\
+ \rho_{12}\sigma_1\sigma_2 \left[ \frac{1 - e^{-a\tau}}{a} - \frac{1 - e^{-2a\tau}}{2a} \right] + \rho_{13}\sigma_1\sigma_3 \left[ \frac{1 - e^{-(k+a)\tau}}{k+a} - \frac{1 - e^{-2a\tau}}{2a} \right] \\
+ \rho_{23}\sigma_2\sigma_3 \left[ \frac{1 - e^{-k\tau}}{k} - \frac{1 - e^{-(k+a)\tau}}{k+a} - \frac{1 - e^{-a\tau}}{a} + \frac{1 - e^{-2a\tau}}{2a} \right] \]

For the specific derivation of (6) please refer to the appendix. Therefore the slope \( \psi \) of the (log) futures term structure

\[ \psi := \ln (F(x_t, y_t, z_t, \tau)) - x_t = (e^{-a\tau} - 1) x_t + B(\tau)y_t + C(\tau)z_t + D(\tau) \]

does decrease in the (log) spot price, which is consistent with the BCSS(1995).

In the risk-neutral measure, the futures price follows,

\[ \frac{dF}{F} = A(\tau)\sigma_1dW_1^Q + B(\tau)\sigma_2dW_2^Q + C(\tau)\sigma_3dW_3^Q \]  

(7)

The futures price in the physical measure follows

\[ \frac{dF}{F} = \left[ A(\tau) \left( \tilde{\theta} - (\tilde{a} - a)x_t \right) + B(\tau)(\tilde{u} - u) + C(\tau)(\tilde{\phi} - \phi) \right] dt \\
+ A(\tau)\sigma_1dW_1^P + B(\tau)\sigma_2dW_2^P + C(\tau)\sigma_3dW_3^P \]  

(8)

Therefore, the SLM model captures three features of commodity futures: 1) in the risk-neutral...
measure, commodity prices are mean-reverting to a stochastic mean, 2) the risk premia of commodity prices decrease in commodity prices and 3) in the physical measure, commodity prices do not revert to a constant level.

If \( u = \hat{u} = \sigma_2 = \rho_{12} = \rho_{23} = 0 \), the model becomes the CC (2005) model with a constant interest rate. If \( u = \hat{u} = \sigma_2 = \hat{a} = a = \rho_{12} = \rho_{23} = 0 \), the model becomes identical to the GS (1990) model.

4 Results

In this section, I first show the model calibration methodology and the data. I then show the calibration results, using oil and copper as examples, and the out-of-sample tests.

4.1 Model Calibration

One of the difficulties in the calibration of our model is that the three factors are not directly observable. Several calibration methodologies have been proposed to solve this problem, such as the efficient method of moments (Gallant and Tauchen 1996), maximum likelihood estimation (i.e., the Chen and Scott 1993 method), and the Kalman filter method. Duffee and Stanton (2004) compare these methods and conclude that the Kalman filter is the best method among those three, especially when the model is complicated. Maximum likelihood estimation produces strongly biased parameters when modeling complex dynamics and even the efficient method of moments is not acceptable. Therefore, in this paper, we specify our model in a state-space form and use the Kalman filter to calibrate the model.\(^6\)

The state-space form normally consists of a transition equation and a measurement equation. The transition equation shows the stochastic process of the data-generating process. Thus, the transition equation in the model should be the discrete version of (5). The measurement equation relates the time series of multivariate observable variables to an unobservable vector of state

\(^6\)Refer to Harvey (1989) for the details of the Kalman filter.
variables (factors \(x, y, z\)). The measurement equation is obtained using (6) with uncorrected noises taking account of the pricing errors. These errors may be caused by bid-ask spreads, the non-simultaneity of the observations, etc.

To describe the transition and measurement equations in greater detail, suppose the data are sampled in an equal interval: \(t_n, n = 1, \ldots, T\). Let \(\Delta = t_{n+1} - t_n\), be the interval between two observations and \(X = (X_1, \ldots, X_T)\) and \(Z = (Z_1, \ldots, Z_T)\) be the total latent state variables and observations (from 1 to \(T\)). \(X_n = [x_n, y_n, z_n]\) represents the vector for the state variables at time \(t_n\), and \(Z_n = [F_1, \ldots, F_{22}]^T\) represents the six observations of each time \(n\). Therefore, the transition equation is specified as

\[
X_{n+1} = HX_n + L + \omega_n, \tag{9}
\]

where

\[
H = \begin{bmatrix}
1 - \tilde{a}\Delta & a\Delta & -\Delta \\
0 & 1 & 0 \\
0 & 0 & 1 - k\Delta
\end{bmatrix}
\]

\[
L = \Delta \begin{bmatrix}
\bar{\theta} & \bar{u} & \bar{\phi}
\end{bmatrix}^T.
\]

Note that since the sample interval in the data is relatively short, for the sake of parsimony we here follow Schwartz (1997) and use the Euler discretization in finding the transition equation. It is certainly feasible to obtain the exact solutions, given the affine structure of the model.

The measurement equation is obtained from the futures pricing formula,

\[
Z_n = V_nX_n + U_n + \varepsilon_n, \tag{10}
\]
where

\[ V_n = \begin{bmatrix}
A(\tau_{F1}) & B(\tau_{F1}) & C(\tau_{F1}) \\
\vdots & \vdots & \vdots \\
A(\tau_{F22}) & B(\tau_{F22}) & C(\tau_{F22})
\end{bmatrix}, \]

\[ U_n = \begin{bmatrix}
D(\tau_{F1}) & \cdots & D(\tau_{F22})
\end{bmatrix}^\top, \]

where \( \tau_i \) denotes the time to maturity of contract \( i \), \( \omega_n \) and \( \varepsilon_n \) are zero-mean random Gaussian noise vectors at time \( t_n \) with their respective variance-covariance matrices \( \Phi \) and \( \Sigma \). The variances of pricing errors are denoted by \( \xi_i^2 \):

\[ \Phi = \Delta \begin{bmatrix}
1 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\
\rho_{12}\sigma_1\sigma_2 & 1 & \rho_{23}\sigma_2\sigma_3 \\
\rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & 1
\end{bmatrix} \]

\[ \Sigma = \begin{bmatrix}
\xi_{F1}^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \xi_{F22}^2
\end{bmatrix}. \]

To obtain optimal estimates of the model parameters I employ the standard estimation-maximization (EM) algorithm. The optimization is repeated using several different initial parameter values in order to avoid the risk of missing the global optimum.

### 4.2 The Data

The dataset consists of crude oil and copper futures contracts. Weekly data\(^7\) are used from Jan.03 1998 to Mar.01 2006 (425 observations for each commodity), and the futures prices of WTI crude oil (CL) and high-grade copper (HG) are obtained from the NYMEX.\(^8\) Table 1 contains the summary statistics for commodity prices and returns. The time to maturity ranges from 1 month to 22

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\(^7\)Data are sampled every five business days.

\(^8\)Note that the oil and copper data are from the NYMEX and COMEX divisions, respectively.
Table 1:

Mean and standard deviation of oil and copper futures returns, weekly data are used from Jan.03 1998 to Mar.01 2006, futures prices of WTI crude oil and high-grade copper (HG) are obtained from the NYMEX. $F_n$ is denoted as the $n^{th}$ contract closest to maturity, $n$ also roughly denotes the time to maturity (in monthly units).

<table>
<thead>
<tr>
<th></th>
<th>$F_1$</th>
<th>$F_5$</th>
<th>$F_9$</th>
<th>$F_{13}$</th>
<th>$F_{17}$</th>
<th>$F_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (weekly return %)</td>
<td>0.2921</td>
<td>0.2931</td>
<td>0.2944</td>
<td>0.2953</td>
<td>0.2955</td>
<td>0.2950</td>
</tr>
<tr>
<td>Std (weekly return %)</td>
<td>5.0084</td>
<td>3.8115</td>
<td>3.1886</td>
<td>2.8272</td>
<td>2.6096</td>
<td>2.4532</td>
</tr>
<tr>
<td>Copper</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (weekly return %)</td>
<td>0.2525</td>
<td>0.2399</td>
<td>0.2278</td>
<td>0.2169</td>
<td>0.2056</td>
<td>0.1937</td>
</tr>
<tr>
<td>Std (weekly return %)</td>
<td>2.8552</td>
<td>2.7053</td>
<td>2.5528</td>
<td>2.4687</td>
<td>2.4419</td>
<td>2.4585</td>
</tr>
</tbody>
</table>

months for both oil and copper contracts. I denote $F_n$ as the $n^{th}$ contract closest to maturity; e.g., $F_1$ is the futures contract that is closest to maturity. Since the maturities are in consecutive calendar months, $n$ also roughly denotes the time to maturity (in monthly units). In this paper, I use six time series for oil and copper – $F_1, F_5, F_9, F_{13}, F_{17}$ and $F_{22}$ contracts.

4.3 Results

Results are shown in Table 2. Nearly all risk-neutral parameters are significant, which shows that the SLM model can model the futures dynamics quite well. One observes three common conclusions for both oil and copper:

1. The parameter $a$ is significantly positive, which shows that the spot price does revert to its long-run mean in the risk-neutral measure, which is consistent with BCSS (1995).

2. $\hat{a} > a$ for both oil and copper (refer to (4)), which is consistent with the empirical finding that the risk premia of spot prices depend negatively on log-spot prices. It also indicates that the speed of mean-reversion in the physical measure is stronger than it is in the risk-neutral measure. The risk premium of the $y$ factor $\lambda_y$ is 0.512 and 0.488 for oil and copper, respectively; the risk premium of the $z$ factor is $-0.412$ and $-1.073$ for oil and copper but is highly insignificant.
3. $\sigma_2$ are significant which proves that the long-run mean $y$ factor is significantly stochastic.

To statistically test whether our model is better than both the CC (2005) model with a constant interest rates and the GS (1990) model, we perform likelihood ratio tests. The results for these two tests, shown in table 3, do inform that the performance of the SLM model is superior to both the CC (2005) model with a constant interest rate and the GS (1990) model.\(^9\) Moreover, there is a significant positive correlation between the $x$ and $y$ factors, which shows that the spot price and its long-run mean tend to move together.

For oil, Figure 2 shows the log-spot price $x$ and the long-run mean factor $y$; it is clear that the $x$ factor fluctuates around the $y$ factor. The half-life of the spot price’s reverting to its long-run mean $y$ is around 8 months. After 2000, the long-term factor $y$ does run up significantly, which shows that investors do expect a permanent shift in the oil price. The convenience yield process is shown in Figure 3. The graph is obtained by using the estimated latent state variables and then calculating the implied convenience yield. The magnitude of the convenience yield is somewhere above 40%, which is consistent with Figure 5 in CC(2005). The large convenience yields correspond to dates with strong backwardation. Because the volatility of spot price (the $x$ factor) $\sigma_1$ is about 50% higher than the volatility of the long-run mean (the $y$ factor), Figure 2 shows the long-run mean of oil has been slowly but steadily increasing, while the log-spot price has been quite volatile. From the small pricing error $\xi_{F1}$ to $\xi_{F22}$, one can see that the SLM model fits the futures prices quite well.

Figure 4 shows the log-spot price $x$ fluctuates around the long-run mean factor $y$ for copper. The half-life of the mean reversion of the spot price to its long-run mean $y$ is around 5 months. The convenience yield of copper is quite significant; this is shown in Figure 5. Note that after 2004, the copper prices dramatically increase and that in the meantime the futures are in deep backwardation, showing that the net demand of copper dramatically increases after 2004 and thus

\(^9\)In the calibration of the CC(2005) with constant interest rates and Gibson-Schwartz (1990) models, I use the same calibration procedure used by the SLM model with $u = \tilde{u} = \sigma_2 = \rho_{12} = \rho_{23} = 0$ and $u = \tilde{u} = \sigma_2 = \tilde{\sigma} = a = \rho_{12} = \rho_{23} = 0$, respectively. By doing this, I thus avoid undesirable potential different numerical setups of the three models during the maximization procedure.
Table 2:
The parameters for oil and copper contracts.

<table>
<thead>
<tr>
<th>variables</th>
<th>Oil</th>
<th>Copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}$</td>
<td>1.2930*</td>
<td>1.8214*</td>
</tr>
<tr>
<td></td>
<td>(0.3362)</td>
<td>(0.1309)</td>
</tr>
<tr>
<td>$a$</td>
<td>1.0252*</td>
<td>1.5917*</td>
</tr>
<tr>
<td></td>
<td>(0.0993)</td>
<td>(0.1021)</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>0.1407</td>
<td>0.0947</td>
</tr>
<tr>
<td></td>
<td>(0.0772)</td>
<td>(0.0630)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.0274*</td>
<td>0.0070*</td>
</tr>
<tr>
<td></td>
<td>(0.0100)</td>
<td>(0.0022)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-0.1138</td>
<td>-0.4027</td>
</tr>
<tr>
<td></td>
<td>(0.6258)</td>
<td>(2.3290)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.0085</td>
<td>-0.1605</td>
</tr>
<tr>
<td></td>
<td>(0.6181)</td>
<td>(2.3279)</td>
</tr>
<tr>
<td>$k$</td>
<td>0.6022*</td>
<td>1.5099*</td>
</tr>
<tr>
<td></td>
<td>(0.1027)</td>
<td>(0.0971)</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.3544*</td>
<td>0.2072*</td>
</tr>
<tr>
<td></td>
<td>(0.0125)</td>
<td>(0.0072)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.2211*</td>
<td>0.1799*</td>
</tr>
<tr>
<td></td>
<td>(0.0144)</td>
<td>(0.0062)</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>0.2920*</td>
<td>0.2258*</td>
</tr>
<tr>
<td></td>
<td>(0.0224)</td>
<td>(0.0119)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.1809</td>
<td>1.1608</td>
</tr>
<tr>
<td></td>
<td>(0.9503)</td>
<td>(1.3887)</td>
</tr>
<tr>
<td>$\rho_{12}$</td>
<td>0.5141*</td>
<td>0.8384*</td>
</tr>
<tr>
<td></td>
<td>(0.0563)</td>
<td>(0.0273)</td>
</tr>
<tr>
<td>$\rho_{13}$</td>
<td>0.1622</td>
<td>-0.1909*</td>
</tr>
<tr>
<td></td>
<td>(0.1081)</td>
<td>(0.0535)</td>
</tr>
<tr>
<td>$\rho_{23}$</td>
<td>0.6569*</td>
<td>0.2314*</td>
</tr>
<tr>
<td></td>
<td>(0.0725)</td>
<td>(0.0493)</td>
</tr>
<tr>
<td>$\xi_{F1}$</td>
<td>0.0268*</td>
<td>0.0017*</td>
</tr>
<tr>
<td></td>
<td>(0.0071)</td>
<td>(0.0002)</td>
</tr>
<tr>
<td>$\xi_{F5}$</td>
<td>0.0001*</td>
<td>0.0040*</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>$\xi_{F9}$</td>
<td>0.0024*</td>
<td>0.0000*</td>
</tr>
<tr>
<td></td>
<td>(0.0004)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>$\xi_{F13}$</td>
<td>0.0000*</td>
<td>0.0020*</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0005)</td>
</tr>
<tr>
<td>$\xi_{F17}$</td>
<td>0.0020*</td>
<td>0.0000*</td>
</tr>
<tr>
<td></td>
<td>(0.0004)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>$\xi_{F22}$</td>
<td>0.0001*</td>
<td>0.0065*</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0017)</td>
</tr>
</tbody>
</table>

Loglikelihood | 8794.0 | 9440.2 |

Numbers in brackets are standard errors. *denotes significance in 95% confidence level.
Table 3:
Likelihood ratio statistics for the SLM model and 1) the CC(2005) model with constant interest rates, 2) the Gibson-Schwartz model. 1% significant levels are 15.1 and 18.5 for 1) and 2) with degree of freedom of 5 and 7 respectively.

<table>
<thead>
<tr>
<th>variables</th>
<th>Oil</th>
<th>Copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = \hat{u} = \sigma_2 = \rho_{12} = \rho_{23} = 0$</td>
<td>2445.8</td>
<td>1646.4</td>
</tr>
<tr>
<td>$u = \hat{u} = \sigma_2 = a = \hat{a} = \rho_{12} = \rho_{23} = 0$</td>
<td>2741.2</td>
<td>1683.2</td>
</tr>
</tbody>
</table>

Figure 2: The log-spot price and the long-run mean factors for oil estimated from the SLM model.

results in a large convenience yield.\textsuperscript{10} In the meanwhile, we also see an increased convenience yield volatility in copper after 2004.\textsuperscript{11} Also, we see that the long-run mean of copper is quite close to the log-price before 2004 but goes much higher afterwards. This means that market investors expect that the long-run prices move up after 2004. This could be caused by the fundamental shifts of demand and supply in copper market; it might also due to the financialization of the commodity market; for details, refer to Tang and Xiong (2009). Moreover, the small pricing errors, $\xi_{F1}$ to $\xi_{F22}$, suggest that the SLM model fits the copper futures prices quite well.

\textsuperscript{10}I also compare the implied convenience yield from this model with that implied from 1-month and 5-month futures, this two time series are very close to each other.

\textsuperscript{11}This is consistent with Liu and Tang (2009) in that high convenience yield is usually accompanied by a high convenience yield volatility.
Figure 3: The convenience yield of oil futures from the SLM model.

Figure 4: The x and y factors for copper futures.
4.4 Out of Sample Tests

In the section, I do an out-of-sample test between the SLM, CC and GS models. The test period is one year from Mar.01 2006 to Mar. 01 2007 (52 observations for each commodity) using futures data for oil (1, 5, 9, 13, 17, 22 months) and copper (1, 3, 5, 7, 9, 11 months). Assuming that the model parameters are unchanged with time, I run two tests to compare the goodness-of-fit and prediction performances of three models. For the first test, I run a Kalman filter to obtain the current latent variables (x, y and z factors) in each week and then calculate the pricing errors of the futures term structures. For the second test, I forecast latent factors using information up to the past week and then use the forecasted factors to price the futures term structures. I compute the root of the mean squared errors (RMSE) and the mean squared prediction errors (MSPE) for all futures with different maturities. Table 4 shows the statistics. The table shows that the SLM model performs much better than the CC and GS models in fitting futures prices, but only does a marginally better job in forecasting.
Table 4:  
**Out of sample tests for SLM, CC and GS models.** For the first 
test, Kalman filter is performed to obtain the current latent variables. For the second test, latent 
factors are forecasted using information up to the past week.

<table>
<thead>
<tr>
<th></th>
<th>SLM</th>
<th>CC</th>
<th>GS</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE– Oil test1</td>
<td>0.0048</td>
<td>0.0093</td>
<td>0.0097</td>
</tr>
<tr>
<td>MSPE– Oil test2</td>
<td>0.0300</td>
<td>0.0310</td>
<td>0.0325</td>
</tr>
<tr>
<td>RMSE– Copper test1</td>
<td>0.0024</td>
<td>0.0080</td>
<td>0.0076</td>
</tr>
<tr>
<td>MSPE– Copper test2</td>
<td>0.0517</td>
<td>0.0533</td>
<td>0.0533</td>
</tr>
</tbody>
</table>

5  **Application – Option Pricing**

Black (1976) has developed a standard model for valuing European futures options when the 
underlying has a log-normal distribution. The futures price in the SLM model also follows a log-
normal distribution, so the option valuation formula should be close to that proposed by Black 
(also refer to (7)). We denote current time as $t$, the futures maturity as $T$, the option maturity 
as $R$, the call option value on the futures contract as $C(t, R)$, the value at time $t$ of a zero-coupon 
bond with maturity at time $R$ as $B(t, R)$ and the strike price of the option as $\Pi$. Then the value 
of the European call under the SLM model’s log-normal distribution is the Black formula with a 
modified volatility, i.e.,

$$C(t, R) = B(t, R) \left[ F(t, T)N(d_1) - \Pi N(d_2) \right]$$

where $d_1 := \frac{\ln(F(t, T)/\Pi) + 0.5v^2}{v}$, $d_2 := d_1 - v$, and $N$ denotes the cumulative standard normal distribu-
tion function. $v^2$ is the integral of the return variance of futures from $t$ to $R$, which is given 
by:
\[ v^2(t, R, T) = \frac{\sigma^2}{2} \left( e^{-2a(T-R)} - e^{-2a(T-t)} \right) + \frac{\sigma^2}{2} \left( R - t \right) - \frac{2(1-e^{-a(T-R)} - e^{-a(T-t)})}{a} + \frac{e^{-2a(T-R)} - e^{-2a(T-t)}}{2a} \]

\[ + \frac{\sigma^2}{(k-a)^2} \left( e^{-2k(T-R)} - e^{-2k(T-t)} \right) - \frac{2(1-e^{-a+k}(T-R) - e^{-a+k}(T-t))}{a+k} + \frac{e^{-2a(T-R)} - e^{-2a(T-t)}}{2a} \]

\[ + \frac{2 \rho_1 \sigma_1 \sigma_2}{k-a} \left( e^{-a(T-R)} - e^{-a(T-t)} \right) - \frac{e^{-2a(T-R)} - e^{-2a(T-t)}}{2a} \]

\[ + \frac{2 \rho_3 \sigma_2 \sigma_3}{k-a} \left( e^{-k(T-R)} - e^{-k(T-t)} \right) - \frac{e^{-2a(T-R)} - e^{-2a(T-t)}}{2a} \]

\[ + \frac{2 \rho_3 \sigma_2 \sigma_3}{k-a} \left[ \frac{e^{-a(T-R)} - e^{-a(T-t)}}{a} + \frac{e^{-2a(T-R)} - e^{-2a(T-t)}}{2a} \right] \]

Note that \( v \) is the only input for calculating option prices. The value of a European put option can be calculated by put-call parity. Note that in reality, options in futures markets are normally American style. The difference between the European and American options is the early exercise premium. However, the determination of the early exercise premium is beyond the scope of this paper. I refer interested readers to Barone-Adesi and Whaley (1987), where early exercise premium of American options on futures is investigated specifically.

To illustrate the difference between the SLM, CC and GS models in pricing options, Figure 6 shows the option prices given a constant futures maturity \((T - t = 5 \text{ years})\) and different option times to maturity \(R - t\). The option values are calculated from the oil dataset mentioned in subsection 4.2 using the three models—GS, CC and SLM. They are quite different for each other. For example, for a three-month at-the-money option with a futures price of 100, the option prices are, 6.27, 5.35, and 5.71 for the SLM, CC and GS models, respectively. Note that the option value in the CC model is less than that in the GS and SLM models because the spot price in the CC model is mean-reverting to a constant level.
6 Conclusion

In this paper, I first discuss the mean-reversion of commodity prices, which relates to three facts: 1) in the risk-neutral measure, commodity prices are mean reverting but do not revert to a constant mean, 2) risk premia decrease in commodity prices and 3) in the physical measure, commodity prices do not revert to a constant mean. All of these facts suggest that commodity prices should revert to a time-varying long-run mean. I also propose a reduced-form model that includes the stochastic long-run mean (SLM) factor. Likelihood-ratio and out-of-sample tests show that the SLM model performs better than those not incorporating the stochastic long-run mean factor (such as the CC and GS models). In addition, for oil and copper, I find that the long-run mean tends to move together with spot prices, which indicates that investors do shift their forecasts of futures prices with time. As an application, the SLM model is used to price options on futures, which have different values with those in the CC and GS models.
References


Appendix – Futures Pricing:

The futures price should then follow the following Feynman-Kac equation

\[
\frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial x} (ay_t - ax_t - z_t) + \frac{\partial F}{\partial y} u + \frac{\partial F}{\partial z} (\phi - k z_t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma_1^2 + \frac{1}{2} \frac{\partial^2 F}{\partial y^2} \sigma_2^2 + \frac{1}{2} \frac{\partial^2 F}{\partial z^2} \sigma_3^2 
\] 

\[
+ \frac{\partial^2 F}{\partial x \partial y} \rho_{12} \sigma_1 \sigma_2 + \frac{\partial^2 F}{\partial x \partial z} \rho_{13} \sigma_1 \sigma_3 + \frac{\partial^2 F}{\partial y \partial z} \rho_{23} \sigma_2 \sigma_3 
\] 

(13)

We guess that the futures price \( F(x_t, y_t, z_t, \tau) := F(t, T) \) follows

\[
F(x_t, y_t, z_t, \tau) = \exp \left[ A(\tau)x + B(\tau)y + C(\tau)z + D(\tau) \right] 
\] 

(15)

with \( A(0) = 1, B(0) = 0, C(0) = 0 \), and \( D(0) = 0 \). \( A(\tau), B(\tau), C(\tau), D(\tau) \) satisfy the following ordinary differential equations,

\[
A' = -aA \\
B' = aA \\
C' = -A - Ck \\
D' = Bu + C\phi + \frac{1}{2} \left[ A^2 \sigma_1^2 + B^2 \sigma_2^2 + C^2 \sigma_3^2 + 2 \rho_{12} \sigma_1 \sigma_2 AB + 2 \rho_{13} \sigma_1 \sigma_3 AC + 2 \rho_{23} \sigma_2 \sigma_3 BC \right]
\]

The solutions of \( A(\tau), B(\tau), C(\tau), D(\tau) \) are shown in the main text.