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A Stochastic Simulation Approach to Model Selection for Stochastic Volatility Models

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Stochastic volatility models have been widely appreciated in empirical finance such as option pricing, risk management, etc. Recent advances of Markov chain Monte Carlo (MCMC) techniques made it possible to fit all kinds of stochastic volatility models of increasing complexity within Bayesian framework. In this article, we propose a new Bayesian model selection procedure based on Bayes factor and a classical thermodynamic integration technique named path sampling to select an appropriate stochastic volatility model. The performance of the developed procedure is illustrated with an application to the daily pound/dollar exchange rates data set.

Keywords Bayes factor; Financial time series; Model selection; Path sampling; Stochastic volatility models.

Mathematics Subject Classification C11; C12; G12.

1. Introduction

In empirical finance literature, the stochastic volatility (SV) models have been regarded as an efficient alternative for ARCH-type models to describe financial volatility over time. In recent years, SV models have been widely applied in empirical finance such as option pricing, risk management, etc. Compared to ARCH-type models, their superiority mainly lies in that one more white noise process is added to assess variation in the underlying volatilities dynamics for latent volatilities; see Ghysels et al. (1996), Meyer and Yu (2000), Shephard (2005), etc. Analysis of stochastic volatility models has provoked a lot of interests of researchers. Various estimation approaches have been proposed for analyzing SV models, such as simulated maximum likelihood inference (Danielsson, 2001), quasi-maximum likelihood inference (Harvey et al., 1994), efficient method of moments (Gallant et al., 1997), and Bayesian inference (Chib et al., 2002; Jacquier et al., 1994; Kim et al., 1998; Meyer and Yu, 2000; Shephard and Pitt, 1977; Yu, 2005 etc).

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In the analysis of SV models, model comparison is one of the most important issues, for example, selecting the appropriate volatility model for value-at-risk, etc. When fitting a particular financial times series data set, we are not only concerned about which model is better between two competitive models, but also which model is the best among the models compared. In recent years, due to the advances of Markov Chain Monte Carlo (MCMC) methods, it becomes possible to fit all kinds of stochastic volatility models of increasing complexity within Bayesian framework. Therefore, in the recent literatures, Bayesian model selection approaches have aroused many interests of researchers. Kim et al. (1998) and Chib et al. (2002) employed the marginal likelihood approach of Chib (1995) to estimate Bayes factor for comparing various SV models. Raggi and Bordignon (2006) also used Bayes factor for comparing SV models, where the method of Chib and Jeliazkov (2001) instead of Chib (1995) is employed to estimate Bayes factor. These approaches for calculating Bayes factor require the marginal likelihoods, a marginalization over the parameter and latent variable vectors in each model. Unfortunately, the number of unknown parameters and latent volatilities in SV models is large (exceeding the number of observations). It remains a computation-intensive task and is not a user-friendly tool for practitioners. Thus, Berg et al. (2004) recommended the use of deviance information criterion (DIC), proposed by Spiegelhalter et al. (2002), to replace Bayes factor for SV models selection purpose. Their method can be easily implemented using a free and reliable software package named WinBUGS (Spiegelhalter et al., 2003).

Although DIC is a convenient tool for model comparison, it differs in at least one important aspect from the Bayes factor. DIC is a predictive measure, of which the goal is to pick a model that achieves best predictions for future data. In contrast, the goal of Bayes factor model selection is to find the model with the highest posterior model probability, or to identify the most appropriate model to fit the data. In practice, prediction using SV models is not the unique objective. Thus, it is still meaningful to develop some efficient approach to compute Bayes factor for model selection of SV models under Bayesian framework. The main objective of this article is to serve this purpose. We develop an efficient stochastic simulation algorithm for computing Bayes factor to select SV models. The approach is based on a classical thermodynamic integration technique named path sampling (Gelman and Meng, 1998), which is a reliable tool for computing ratio of normalizing constants of probability models. Path sampling has been shown to enjoy the following advantages (Gelman and Meng, 1998; Lee, 2007).

(i) It is a generalization of importance sampling and bridge sampling (Meng and Wong, 1996), and more accurate results can be obtained.
(ii) Instead of computing the ratio, it computes the logarithm scale of the ratio, hence the implementation is generally more stable.
(iii) Its main computational effort lies in drawing random observations from the posterior distributions, hence the implementation is simple.

In what follows, we will show that the proposed procedure also can be simply implemented using WinBUGS with a package named R2WinBUGS (Sturtz et al., 2005).

The rest of this article is organized as follows. In Sec. 2, we give a brief review of Bayesian SV models. In Sec. 3, a procedure based on path sampling is established to compute Bayes factor for SV models comparison. Then we present a simple
account of WinBUGS with R2WinBUGS. Section 4 provides a simulation study and a comparison of the effect of Bayes factor with DIC. Section 5 illustrates an application of the proposed approach to a real data set. Finally, some conclusions and discussions are included in Sec. 6.

2. Stochastic Volatility Models

In this section, we give a simple description of the basic SV model. Given a time series of return $y_t$, $t = 1, 2, \ldots, n$, the basic formulation of SV models consists of two equations: the observation equation and the state equation. The observation equation specifies the conditional distributions of data given unknown states and generally can be expressed as follows:

$$y_t = \exp(h_t/2)u_t, \quad u_t \sim N(0, 1), \quad t = 1, 2, \ldots, n. \quad (1)$$

where $y_t$ is the response variable, $h_t$ is the unobserved state indicating the log-volatility process of $y_t$. The error $u_t$ is independently Gaussian white noise for all $t$. The state equation assumes that the unknown states follow a Markov process and can be described as:

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 1, 2, \ldots, n. \quad (2)$$

Here, state $h_0 \sim N(\mu, \tau^2)$, and $v_t$ is also independently Gaussian white noise and is uncorrelated with $u_t$ for all $t$. The recent years have witnessed various extensions of the above basic SV model. It is impractical to compare all of them. Thus, in this article, we only compare some of them, which are described in Sec. 4, for illustration.

For SV models, the likelihood function involved many unobserved states, hence intractable. Classical parameter estimation procedure is hard to be put into practice. Fortunately, with the aid of MCMC techniques, Bayesian approach provides an appealing scheme. Bayesian analysis is based on the posterior distribution of unobservable components, parameters and latent states. For simplicity, let $p(\theta)$ be the joint prior distribution of $\theta = (\mu, \phi, \tau, \sigma)$ in formula (1) and (2). Denote $y = (y_1, \ldots, y_n)$ and $h = (h_0, h_1, \ldots, h_n)$. Then the complete likelihood of $(\theta, h, y)$ can be expressed as follows:

$$p(\theta, h, y) = p(\theta, h)p(y | \theta, h) = p(\theta) \prod_{t=1}^n p(h_t | h_{t-1}, \theta) \prod_{t=1}^n p(y_t | \theta, h). \quad (3)$$

By Bayes’s theorem, the posterior distribution of $(\theta, h)$ given by the data $y$ is proportional to the complete likelihood, that is,

$$p(\theta, h | y) = \frac{p(\theta, h, y)}{p(y)} \propto p(\theta, h, y). \quad (4)$$

The posterior distribution $p(\theta, h | y)$ can be realized via MCMC techniques utilizing the idea of data-augmentation strategy (Tanner and Wong, 1996). Several different algorithms have been proposed; see Jacquier et al. (1994), Kim et al. (1998), and Chib et al. (2002). Observations obtained from the posterior simulation can be used for statistical inference. Bayesian estimates of $\theta$ and the latent
volatilities $h$ can be obtained easily via the corresponding sampling means of the generated observations. Specifically, let $\{\theta^{(j)}, h^{(j)}, j = 1, 2, \ldots, J\}$ be the efficient random observations generated from the joint posterior distribution $p(\theta, h | y)$ after discarding the burn-in samples of some length. Then the joint Bayesian estimates of $\theta, h$ can be obtained, respectively, as follows

$$
\hat{\theta} = \frac{1}{J} \sum_{j=1}^{J} \theta^{(j)}, \quad \hat{h} = \frac{1}{J} \sum_{j=1}^{J} h^{(j)}.
$$

(5)

Clearly, these Bayesian estimates are consistent estimates of the corresponding posterior means (Geyer, 1992). A consistent estimate of $\text{Var}(\theta | y)$ can be obtained as follows:

$$
\hat{\text{Var}}(\theta | y) = \frac{1}{J-1} \sum_{j=1}^{J} (\theta^{(j)} - \hat{\theta})(\theta^{(j)} - \hat{\theta})^T.
$$

Although Bayesian approach using MCMC techniques is very efficient, even evaluated as one of the best approaches for analyzing SV models (Andersen et al., 1999), it still remains a computationally demanding task for practitioners (Berg et al., 2004). It is well appreciated if some efficient software package can be used to free the researchers from tedious programming and debugging. For this purpose, Meyer and Yu (2000) first introduced the implementation of a Bayesian analysis of SV models using a software named BUGS. They illustrated the ease with which Bayesian posterior computation of SV models can be performed via BUGS. Several later researchers advocated the use of BUGS software for Bayesian SV models in a number of studies; see Berg et al. (2004), Yu (2005), Yu and Meyer (2006), Congdon (2007), among others.

BUGS is a recently developed, user-friendly, freely available software package. Under general conditions, this software can simulate efficient sequences of random samples from the joint posterior distribution of the unknown quantities for reliable Bayesian statistical inference. This saves a lot of time for researchers by avoiding the usually lengthy implementations and debugging of MCMC simulation algorithms. The latest version of BUGS is WinBUGS1.4 which is developed by the medical Research Council (MRC) Biostatistics Unit (Cambridge, UK) and the department of Epidemiology and Public Health of the Imperial College School of Medicine at St Mary’s Hospital (London). It is available free of charge at: http://www.mrc-bsu.cam.ac.uk/bugs/. An introduction to this software can be found in Spiegelhalter et al. (2003). Following their remarks and suggestions, we will show that the proposed model selection procedure for choosing SV models using Bayes factor also can be performed via WinBUGS1.4 combined with a so-called R2WinBUGS (Sturtz et al., 2005) package.

3. Bayesian Model Selection

Bayes factor is a very important statistic in Bayesian literature. For many years, it has served as one of most popular tools for model testing and comparison. Kass and Raftery (1995) gave an excellent review of Bayes factor, including its interpretation, computation, sensitive analysis, and some applications in real scientific problems. It has drawn much attention in recent years; see DiCiccio et al. (1997), Han and Carlin.
In general, suppose that the observable data $y$ have arisen under one of the two nested or non-nested competing models $M_1$ and $M_0$. Let $\pi(M_k)$ be the prior probability densities, $p(y | M_k)$ be the probability densities of $y$ given $M_k$, and $p(M_k | y)$ be the posterior probability densities, respectively, $k = 0, 1$. The choice between $M_1$ and $M_2$, based on Bayes factor, is commonly defined by

$$B_{10} = \frac{p(M_1 | y)/p(M_0 | y)}{\pi(M_1)/\pi(M_0)},$$

which is the ratio of posterior odds of the model $M_k$ to prior odds of $M_k$. According to Bayes' Theorem, it can be shown that

$$p(M_k | y) = \frac{p(y | M_k)\pi(M_k)}{p(y | M_1)\pi(M_1) + p(y | M_0)\pi(M_0)}.$$

Then, we can easily get that

$$B_{10} = \frac{p(y | M_1)}{p(y | M_0)},$$

where $p(y | M_k)$ is generally called marginal likelihood and can be obtained by integrating over the parameter space as follows:

$$p(y | M_k) = \int p(y | \theta_k, M_k)p(\theta_k | M_k)d\theta_k, \quad \theta_k \in \Omega_k, \quad k = 0, 1.$$

Thus, the Bayes factor is essentially the ratio of the marginal likelihoods corresponding to the two competitive models.

According to Kass and Raftery (1995) and Lee (2007), the Bayes factor can be explained as a summary of the evidence provided by the observable data in favor of a statistical model $M_1$ as opposed to the other model $M_0$. It measures how well $M_1$ predicts the data relative to $M_0$. Kass and Raftery (1995) recommended a scale of evidence for interpreting Bayes factor. Miazhynskaia et al. (2006) suggested a slight modification to make the scale of evidence symmetrical. Following their suggestion, we recommend interpreting the resulting statistic based on the following symmetrical scale of evidence; see Table 1.

### 3.1. Path Sampling for Computing Bayes Factor

It is well known that the computation of Bayes factor is non-trivial (Chib, 1995; Chib and Jeliazkov, 2001; DiCiccio et al., 1997; Kass and Raftery, 1995; Lee, 2007; Li, 2007). The comparative study by DiCiccio et al. (1997) pointed out that bridge sampling (Meng and Wong, 1996) is an attractive algorithm for computing Bayes factor. However, Gelman and Meng (1998) showed that path sampling, an extension of the bridge sampling, is expected to obtain more reliable results. In this article, we use path sampling to compute Bayes factor.

In practice, two different ideas may be considered to compute out the Bayes factor. The first one is based on the marginal likelihood. In fact, the powerful path sampling can be conveniently used to compute the marginal likelihood. For SV
models, consider the following class of densities on the same support with a continuous path parameter \( b \in [0, 1] \):

\[
p(\theta, \mathbf{h} | y, b) = \frac{1}{z(b)} p(\theta, \mathbf{h}, y | b) = \frac{1}{z(b)} p(y | \theta, \mathbf{h}, b) p(\theta, \mathbf{h} | b),
\]

(10)

where \( z(t) = p(y | b) \) such that \( z(1) = p(y | 1) = p(y) \) and \( z(0) = p(y | 0) \) which is a predetermined constant. According to Gelman and Meng (1998), we can get

\[
\frac{\partial \log z(b)}{\partial b} = \int \frac{1}{z(b)} \frac{\partial}{\partial b} p(\theta, \mathbf{h}, y | b) d\mathbf{h} d\theta = E_{\theta, \mathbf{h}} \left[ \frac{\partial}{\partial b} \log p(\theta, \mathbf{h}, y | b) \right].
\]

(11)

where the expectation \( E_{\theta, \mathbf{h}} \) is taken with respect to the distribution \( p(\theta, \mathbf{h} | y, b) \). Let

\[
\mathcal{H}(\theta, \mathbf{h}, y, b) = \frac{\partial}{\partial b} \log p(\theta, \mathbf{h}, y | b),
\]

(12)

then

\[
R = \log \frac{z(1)}{z(0)} = \int_0^1 E_{\theta, \mathbf{h}}[\mathcal{H}(\theta, \mathbf{h}, y, b)] db.
\]

(13)

Therefore, the marginal likelihood \( p(y) \) can be obtained from the formula \( p(y) = z(0) \times \exp(R) \).

To approximate \( R \), we follow the idea of Gelman and Meng (1998) and Lee (2007) to numerically evaluate the integral over \( t \) via the trapezoidal rule as in Ogata (1990). Specifically, we first order the values of different \( b^{(i)} \) from fixed grids \( \{ b^{(i)} \}_{i=0}^s \) such that \( b_0 = 0 < b_{(1)} < b_{(2)} < \cdots < b_{(s)} < b_{(s+1)} = 1 \), and estimate \( R \) by

\[
\hat{R} = \frac{1}{2} \sum_{i=0}^s (b_{(i+1)} - b_{(i)}) (\mathcal{H}_{(i+1)} + \mathcal{H}_{(i)}),
\]

(14)

where

\[
\mathcal{H}_{(i)} = \frac{1}{J} \sum_{j=1}^J \mathcal{H}(\theta^{(j)}, h^{(i)}, y, b^{(i)}),
\]

(15)
in which \( \{ \theta^{(j)}, h^{(j)} \}, j = 1, 2, \ldots, J \) are efficient random observations simulated from \( p(\theta, h \mid y, b_{(0)}) \) after discarding some burn-in samples.

The second one is to link the two competitive models with a graph, and then directly compute the estimated Bayes factor rather than computing the marginal likelihood individually. However, when the parameter vector under two models are defined on two different dimensional parameter space, path sampling cannot be directly used to compute the Bayes factor (Chen et al., 2000; Gelman and Meng, 1998). This problem is generally called different dimension problem (Chen et al., 2000). It can be solved by augmenting the lower dimension density into one which has the same dimension as the higher one by introducing a weight function. To illustrate this idea, for \( k = 0, 1 \), denote \( \Omega_k \) as the parameter space of \( \theta_k \) and \( \Omega_q \) as the parameter space of the latent state \( \mathbf{h} \). Without loss of generality, we assume that the dimension of \( \theta_0 \) is less than that of \( \theta_1 \). Let \( \theta_1 = (\theta_0^T, \theta_0^T)^T \), and

\[
q_0(\theta_1, \mathbf{h}) = p(\theta_0, \mathbf{h} \mid \theta_0)p(\theta_0 \mid M_0)w(\theta_{-0} \mid \theta_0), \\
q_1(\theta_1, \mathbf{h}) = p(\theta_1, \mathbf{h} \mid M_1)p(\theta_1 \mid M_1)w(\theta_{-0} \mid \theta_0),
\]

where \( w(\theta_{-0} \mid \theta_0) \) is a completely known weight function satisfying

\[
\int w(\theta_{-0} \mid \theta_0) \, d\theta_{-0} = 1.
\]

Then it is easy to derive that

\[
\int_{\Omega_1 \cup \Omega_0} q_0(\theta_1, \mathbf{h}) \, d\theta_1 \, d\mathbf{h} = p(y \mid M_0), \\
\int_{\Omega_1 \cup \Omega_0} q_1(\theta_1, \mathbf{h}) \, d\theta_1 \, d\mathbf{h} = p(y \mid M_1).
\]

Thus, we may view \( p(y \mid M_0) \) as the normalizing constant of \( q_0(\theta_1, \mathbf{h}) \), and \( p(y \mid M_1) \) as the normalizing constant of \( q_1(\theta_1, \mathbf{h}) \). Now consider a class of densities, \( q(\theta_1, \mathbf{h} \mid b) \), on the same support \( \Omega_1 \cup \Omega_0 \) with a continuous path parameter \( b \in [0, 1] \), such that \( q(\theta_1, \mathbf{h} \mid 1) = q_1(\theta_1, \mathbf{h}) \) and \( q(\theta_1, \mathbf{h} \mid 0) = q_0(\theta_1, \mathbf{h}) \). Then, path sampling can be used to compute the ratio of the two normalizing constant directly. According to Gelman and Meng (1998), we can obtain

\[
\log B_{10} = \log \frac{p(y \mid M_1)}{p(y \mid M_0)} = \int_0^1 E_{\theta_1,b} [\mathcal{U}(\theta_1, \mathbf{h}, y, b)] \, db,
\]

where \( E_{\theta_1,b} \) denotes the expectation w.r.t the posterior distribution \( p(\theta, \mathbf{h} \mid y, b) \) and

\[
\mathcal{U}(\theta_1, \mathbf{h}, y, b) = \frac{d}{db} \log q(\theta_1, \mathbf{h} \mid b).
\]

Again, the numerical method of Ogata (1990) is recommended to evaluate the integration in formula (17).

**Remark 3.1.** In practice, the prior distributions of \( \theta_0 \) under the two competing models are often specified as the same, i.e., \( p(\theta_0 \mid M_0) = p(\theta_0 \mid M_1) \). Hence, the conditional prior distribution \( p(\theta_{-0} \mid \theta_0, M_1) \) can be regarded as the weight function \( w(\theta_{-0} \mid \theta_0) \). Thus, we obtain

\[
q_0(\theta_1, \mathbf{h}) = p(\theta_1, \mathbf{h} \mid y \mid b) = p(\mathbf{h}, y \mid \theta_1, b)p(\theta_1 \mid M_1),
\]

(19)
where \( p(h, y | \theta_1, 1) = p(h, y | \theta_1, M_1), p(h, y | \theta_1, 0) = p(h, y | \theta_0, M_0). \) Then it can be simply obtained
\[
\mathcal{U}(\theta, h, y, b) = \frac{d}{db} \log q(\theta | h, b) = \frac{d}{db} \log p(y | h, \theta_1, b), \tag{20}
\]
which means that \( \mathcal{U}(\cdot) \) is independent with the prior distributions of the parameters.

**Remark 3.2.** To evaluate the marginal likelihood of model \( M_1, p(y | M_1) \), a practical choice for constant \( z_0 \) is the marginal likelihood of its simpler nested model \( p(y | M_0) \), with \( p(\theta_0 | M_0) = p(\theta_0 | M_1) \). This problem can be formulated as a situation discussed in Remark 3.1.

**Remark 3.3.** For non nested models, we can augment the support space from \( \Omega_k \cup \Omega_h \) \( (k = 0, 1) \) to \( \Omega_0 \cup \Omega_1 \cup \Omega_h \), which is outlined as a different dimension problem.

3.2. **Implementation using WinBUGS with R2WinBUGS Package**

It can be observed easily that the application of path sampling to computing Bayes factor for comparing SV models is rather simple. Its main computational effort is to draw random observations from the posterior distributions with some replications. Thus, for practical researchers, the advantage of our proposed method is its simple implementation. Unfortunately, the WinBUGS language is relatively limited and hard to use for effective analysis involved repeated calls, hence cannot serve our purpose. R2WinBUGS is a R package that calls WinBUGS1.4 and exports the results into R (Sturtz et al., 2005). We use this package to implement our Bayesian model comparison procedure.

4. **A Simulation Study**

In this section, we compare the effect between Bayes factor and DIC. As to the basic model shown in Eqs. (1) and (2), we consider a state equation model comparison problem with the same observable equation shown as follows:
\[
\begin{align*}
  h_t &= \tau + (h_{t-1} - \tau) + \sigma v_t, \quad \text{v.s.} \\
  h_t &= \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad \phi \in (0, 1).
\end{align*}
\]

This model comparison problem has strong economical meaning in econometrics. It can be explained as unit root testing problem, that is, \( H_0 : \phi = 1, H_1 : 0 < \phi < 1 \).

In the simulation study, the true model is the basic model where the true parameters are specified as follows:
\[
\phi = 0.96, \quad \tau = 0.0, \quad \sigma^2 = 0.1, \quad t = 1, 2, \ldots, n.
\]

Time series of different lengths, with \( n = 300, 500, \) and \( 1,000 \), were considered. The whole simulation study is based on 100 replications. For Bayesian analysis of SV models, the appropriate prior distribution needs to be specified. According to Meyer and Yu (2000), we may specify the prior distributions as follows:
\[
\tau \sim N(0, 10), \quad \phi \sim \text{Beta}(1, 1), \quad \frac{1}{\sigma^2} \sim \Gamma(10, 1).
\]
In the replications, we simulated 8,000 random samples of which 5,000 are discarded as burn-in samples. The remaining 3,000 effective random samples are used to obtain the estimated Bayes factor and DIC. We record the number of correct decisions for different \( n \) over the 100 replications where the correct decisions are made in the sense that the Bayes factor is greater than 1 and the DIC of the compared models is smaller. The simulation results are reported in Table 2. We can easily find that the Bayes factor outperforms DIC dominantly. The results is not surprising because, as emphasized in Sec. 1, DIC is to select a better predicting model, while Bayes factor chooses a better fitting model.

5. An Empirical Study

For illustration, we use a data set previously analyzed by Harvey et al. (1994), later by Shephard and Pitt (1977), Kim et al. (1998), Durbin and Koopman (2000), Meyer and Yu (2000), etc. The data consist of a time series of daily pound/dollar exchange rates \( \{z_t\} \) from the period 01/10/81 to 28/6/85. The series of interest are \( 946 \) daily mean-corrected returns \( \{y_t\} \), given by the transformation \( y_t = \log(z_t) - \log(z_{t-1}) - \frac{1}{n} \sum_{t=1}^{n} (\log z_t - \log z_{t-1}) \). This data set can be freely obtained in Congdon (2007).

To introduce the proposed Bayesian model comparison procedure based on Bayes factor and path sampling, we consider both of nested and nonnested SV models cases. Firstly, we compare some nested SV models. Denote the basic SV model introduced in Sec. 2 as model \( M_0 \), nested in model \( M_1 \), a simple nested extension in that an additional nonzero mean \( \mu \) is added into the observation equation, that is:

\[
M_1: \quad y_t = \mu + \exp(h_t/2)u_t, \quad u_t \sim N(0, 1),
\]

\[
h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 1, 2, \ldots, n.
\]

Following Kim et al. (1998), Meyer and Yu (2000), and Congdon (2007), the prior distributions are specified as follows:

\[
\mu \sim N(0,0, 10), \quad \tau \sim N(0,10), \quad \phi = 2\phi^* - 1,
\]

\[
\phi^* \sim Beta(20, 1.5), \quad \frac{1}{\sigma^2} \sim Gamma(2.5, 0.025).
\]

This type of prior specification may be regarded as an informative but reasonably flat prior distributions. To apply the path sampling to computing the Bayes factor for comparing \( M_0 \) and \( M_1 \), a simple intermediate model can be expressed as follows:

\[
M_1^* : \quad y_t = b \times \mu + \exp(h_t/2)u_t, \quad u_t \sim N(0, 1),
\]
Following Chib et al. (2002), the prior distribution of the added parameters may be asymmetric effect, which can be modeled by a correlation between transient movements (Chib et al., 2002; Meyer and Yu, 2000). In empirical finance literature, jump models are quite popular in continuous time models of financial asset pricing. Hence, next, we consider another extension: where AR(2) process is further considered in the state equation:

\[ h_t = \tau + \phi_1(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, \sigma^2), \quad t = 1, 2, \ldots, n. \]

Clearly, it holds that \( M_0^* = M_0 \) and \( M_1^* = M_1 \). It can be simply derived that \( \Theta(h, y, b) = \sum_{i=1}^n h(y_i - h \mu) \exp(-h_i) \). Now we take grids in \([0, 1]\) and set \( J = 3,000 \) to compute Bayes factor, discarding 5,000 burn-in random samples after convergence diagnostic. The burn-in cutoff point is determined using CODA (Convergence Diagnostic and Output Analysis Software for Gibbs sampling output) under R software in one test run, see Best et al. (1995). Then, using the proposed approach in Sec. 3, we can get the estimated log-Bayes factor \( \hat{B}_{10} = -2.94 \). According to the criterion of selecting models given in Sec. 3, Bayes factor supports model \( M_0 \), and we conclude that \( M_0 \) is better than \( M_1 \) for this data.

Secondly, we compare model \( M_0 \) with another nested extension, in which an AR(2) process is further considered in the state equation:

\[
M_2: \quad y_t = \mu + \exp(h_t/2)u_t, \quad u_t \sim N(0, 1); \quad h_0, h_1 \sim N(\tau, \sigma^2), \\
h_t = \tau + \phi_1(h_{t-1} - \tau) + \phi_2(h_{t-2} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 3, 4, \ldots, n.
\]

The prior distribution is specified as: \( \phi_1 = 2\phi_1^* - 1, \phi_1^* \sim Beta(20, 1.5) \), \( k = 1, 2 \), and the other parameters are specified the same as in model \( M_0 \). Similarly, we find an intermediate model, \( M_0^* \), to link these two models:

\[
M_0^*: \quad y_t = \exp(h_t/2)u_t, \quad u_t \sim N(0, 1); \quad h_0, h_1 \sim N(\tau, \sigma^2), \\
h_t = \tau + \phi_1(h_{t-1} - \tau) + b \times \phi_2(h_{t-2} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 3, 4, \ldots, n.
\]

Similarly, we can easily obtain that \( \hat{B}_{20} = -7.03 \), hence \( M_0 \) might be better than model \( M_2 \).

In empirical finance literature, jump models are quite popular in continuous time models of financial asset pricing. Hence, next, we consider another extension: a jump component is incorporated into the observation equation to allow for large and transient movements (Chib et al., 2002; Meyer and Yu, 2000).

\[
M_3: \quad y_t = s_t q_t + \exp(h_t/2)u_t, \quad u_t \sim N(0, 1), \\
h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 1, 2, \ldots, n,
\]

where \( q_t \) is an ordinary Bernoulli trial with \( P(q_t = 1) = \pi \), and \( \ln(1 + s_t) \sim N(-\eta^2/2, \eta^2) \). \( s_t q_t \) can be viewed as a discretization of a finite activity Lévy process. Following Chib et al. (2002), the prior distribution of the added parameters may be specified as: \( \pi \sim Beta(2, 100), \ln(\eta) \sim N(0.05, 0.004) \). To compare \( M_3 \) and \( M_0 \), we can use the following intermediate model:

\[
M_0^*: \quad y_t = b \times s_t q_t + \exp(h_t/2)u_t, \quad u_t \sim N(0, 1), \\
h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1), \quad t = 1, 2, \ldots, n.
\]

The estimated log-Bayes factor \( \hat{B}_{10} = 1.144 \), hence model \( M_3 \) might be better than \( M_0 \).

In financial time series, it is often observed that there is a leverage or an asymmetric effect, which can be modeled by a correlation between \( u_t \) and \( v_{t+1} \); see
Meyer and Yu (2000), Jacquier et al. (1994), and Berg et al. (2004). In Model 4, we take into account this leverage or asymmetric effect, which is expressed as

$$M_4 : \quad y_t = s_t q_t + \exp(h_t/2) u_t, \quad h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad t = 1, 2, \ldots, n$$

$$u_t \sim N(0, 1), \quad v_t \sim N(0, 1), \quad \text{cov}(u_t, v_{t+1}) = \rho.$$ 

As pointed by Meyer and Yu (2000), this model can be alternatively specified by

$$M_4 : \quad y_t = s_t q_t + \frac{\rho}{\sigma^2} \exp(h_t/2) \{\theta_{t+1} - \tau - \phi(h_t - \tau)\} + \exp(h_t/2) \sqrt{1 - \rho^2} u_t,$$

$$h_{t+1} = \tau + \phi(h_t - \tau) + \sigma v_t, \quad t = 1, 2, \ldots, n - 1.$$ 

Then, to compare $M_3$ and $M_4$, a link model can be expressed as

$$M_5^* : \quad y_t = s_t q_t + \frac{\rho}{\sigma^2} \exp(h_t/2) \{\theta_{t+1} - \tau - \phi(h_t - \tau)\} + \exp(h_t/2) \sqrt{1 - \rho^2} u_t,$$

$$h_{t+1} = \tau + \phi(h_t - \tau) + \sigma v_t, \quad t = 1, 2, \ldots, n - 1.$$ 

The estimated log-Bayes factor $\hat{B}_{34} = 29.56$, hence model $M_4$ might be better than $M_3$.

In the above, we compared the basic model $M_0$ with its some nested extensions. Many empirical studies about financial markets have showed that heavy-tailed distribution can be a good substitute for normal distribution in financial time series. Now, following Berg et al. (2004), we consider one nonnested extension of model $M_0$, where the normal errors are replaced by independent central student $t$ distributions with unknown degrees of freedom. This model can be expressed as follows:

$$M_5 : \quad y_t = \exp(h_t/2) u_t, \quad u_t \sim t(k), \quad h_0 \sim t(k),$$

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1).$$ 

According to Berg et al. (2004), the scale mixture of normal can be used to replace $t$ distribution. As a consequence, $M_5$ can be alternatively specified as:

$$M_5 : \quad y_t = \frac{\exp(h_t/2)}{\sqrt{w_t}} \epsilon_t, \quad \epsilon_t \sim N(0, 1), \quad w_t \sim Gamma\left(\frac{k}{2}, \frac{k}{2}\right),$$

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1).$$

To compare $M_5$ and $M_0$, the intermediate model can be specified as:

$$M_5^* : \quad y_t = \frac{\exp(h_t/2)}{\sqrt{1 - b + b \times w_t}} \epsilon_t, \quad \epsilon_t \sim N(0, 1), \quad w_t \sim Gamma\left(\frac{k}{2}, \frac{k}{2}\right),$$

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \quad v_t \sim N(0, 1).$$ 

The estimated log-Bayes factor $\hat{B}_{30} = 13.96$, which provides strong evidence to support this replacement. As to comparing nonnested models $M_4$ and $M_5$, it need not
compute the Bayes factor again. Notice that $\log B_{54} = \log B_{50} - \log B_{40} = \log B_{50} - (\log B_{30} + \log B_{10})$, we can easily obtain that the estimated log-Bayes factor $\hat{\log} B_{54} = 13.96 - (29.57 + 1.14) = -16.75$. Thus, Bayes factor strongly support $M_4$, which implies that $M_4$ might be better than $M_5$.

### 5.1. Sensitivity Analysis

To investigate the sensitivity of Bayes factor with respect to the prior inputs, following the suggestion of Kass and Raftery (1995), we perturb the hyperparameter values of the prior inputs. We focus on the hyperparameters of the prior distributions of the common parameters $\tau, \phi, \sigma^2$ among the models to be compared. For convenience, we call the above prior inputs Type I prior inputs. The data set is reanalyzed with the following two types of prior inputs, respectively: doubling and halving the hyper-parameters of these common parameters, with the other hyper-parameters kept the same as Type I, named as Type II and III prior inputs, respectively. Results are reported in Table 3. We can find that almost all the results from Type II and III are reasonably close to the results from the original prior inputs, which means that Bayes factor is quite robust to the different prior inputs. From these results, it seems that the Bayes factors support the same conclusion in selecting model $M_4$. Hence, $M_4$ is the best model for this data set among all the compared models. Nonetheless, we should point out that since it is impossible to include all possible models in the comparison, we cannot claim that the selected model is the globally best model fitting this data set.

### 6. Conclusion

SV models have been widely used to describe financial volatility over time in empirical finance. Model comparison is one of the most important issues in the analysis of SV models. We proposed a new Bayesian model comparison procedure based on Bayes factor and path sampling for SV models. It can be observed easily that the proposed model selection procedure for selecting SV models is rather simple. Its main effort is to find an intermediate model to link the models to be compared, and then draw random observations from the posterior distributions. We showed that the developed Bayesian model comparison procedure can be implemented easily without too much efforts using a free but reliable software named WinBUGS combined with a package named R2WinBUGS. Thus, for researchers, an important advantage of our proposed method is the easy programming. Anyone interested in this article can obtain the free programming codes using R and WinBUGS through contacting with the authors. At last, we need point out that it is still ambitious to say that path sampling is better than other methods such as the marginal likelihood method developed in Chib (1995) and Chib

<table>
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<th>Prior</th>
<th>$\log B_{10}$</th>
<th>$\log B_{20}$</th>
<th>$\log B_{30}$</th>
<th>$\log B_{43}$</th>
<th>$\log B_{50}$</th>
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<td>Type II</td>
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<td>1.03</td>
<td>33.28</td>
<td>16.62</td>
</tr>
</tbody>
</table>
and Jeliazkov (2001) or others. To answer this question, further investigations into different methods for estimating Bayes factor are necessary, which may be a future research topic.

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References


